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TIME DOMAIN APPROACH TO GAS PULSATION MODELING

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ABSTRACT

The purpose of this paper is to introduce a time domain approach to the manifold system. Four-pole parameters are usually used to solve the continuous wave equation of the system. The four-pole matrix is a very convenient concept for the analysis of a composite acoustic system, but it cannot be easily used to obtain the transient response to an excitation. The four-pole method and the time domain approach are both presented in this paper. The validity of the procedure is proven for a simple case, which is a pipe connected to a small cavity, a volume-neck type manifold. Figure 1 shows the geometry of the problem. Also, the way of converting equivalent viscous damping from model to model is demonstrated by using the modal series expansion to the wave equation.

BASIC MODELS

In this section, several methods are demonstrated for solving a simple acoustic system by a volume-neck type manifold (Figure 1).

Lumped Parameter Model in Four-Pole Form

The volume-neck type manifold can be divided into a small cavity and a short lumped parameter pipe (Figure 1). First, let us consider a small cavity. The acoustic pressure-volume relationship of a small cavity is,

$$P=-\rho C_0^2\frac{dV}{V}, \tag{1}$$

where C_0 is the speed of sound, ρ_0 is average density of the gas in the cavity, and V is volume of the cavity. Therefore,

$$P=-\frac{\rho_0\,C_0^2}{V}\left[\int^t Q_L dt-\int^t Q_0 dt\right]=\frac{\rho_0\,C_0^2}{j\omega V}\left(Q_o-Q_L\right). \tag{2}$$

The cavity can be approximated as a lumped parameter acoustic element. By rearranging equation (2),

$$P=P_0=P_L\ ,\ Q_o=\frac{P_L}{\rho_0 C_0^2}j\omega V+Q_L. \tag{3,4}$$

Rewriting equations (3) and (4) in matrix form, the four-pole equation is obtained for a lumped parameter volume as,

$$\begin{Bmatrix}Q_0\\P_0\end{Bmatrix}=\begin{bmatrix}1 & \dfrac{j\omega V}{\rho_0 C_0^2}\\ 0 & 1\end{bmatrix}\begin{Bmatrix}Q_L\\P_L\end{Bmatrix}. \tag{5}$$

Now the neck which is connected to the cavity is considered. If a gas is contained in a relatively short pipe which is connected to a volume, the gas vibrates as if it were an incompressible mass connected to a spring. The equilibrium equation of this mass is, referring to Figure 1,

$$(P_{L1}-P_{L2})A-D\dot{\xi}=\rho_0 AL\ddot{\xi}, \tag{6}$$

where D is the damping coefficient and A is the cross-sectional area of the pipe. The volume velocity is the same at both ends of the neck because the gas is assumed to be incompressible. Since $A\dot{\xi}=Q_{L1}=Q_{L2}$, equation (6) becomes,

$$P_{L1}=\left(\frac{D}{A^2}+j\omega\frac{\rho_0 L}{A}\right)Q_{L2}+P_{L2}. \tag{7}$$

Therefore, the four-pole equation becomes,

$$\begin{Bmatrix}Q_{L1}\\P_{L1}\end{Bmatrix}=\begin{bmatrix}1 & 0\\ \dfrac{D}{A^2}+j\omega\dfrac{\rho_0 L_1}{A} & 1\end{bmatrix}\begin{Bmatrix}Q_{L2}\\P_{L2}\end{Bmatrix}. \tag{8}$$

The boundary conditions between the cavity and the pipe are, $P_L=P_{L1}, Q_L=Q_{L1}$. (9)

Therefore, the four-pole matrix between P_0, Q_0 and P_{L2},Q_{L2} is,

$$\begin{Bmatrix} Q_0 \\ P_0 \end{Bmatrix} = \begin{bmatrix} 1 & \frac{j\omega V}{\rho_0 C_0^2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{D}{A^2} + j\omega\frac{\rho_0 L_0}{A} & 1 \end{bmatrix} \begin{Bmatrix} Q_{L2} \\ P_{L2} \end{Bmatrix} \quad . \tag{10}$$

The boundary condition at the end of the pipe can be treated in the compressor on the suction side as a pressure release condition, $P_{L2} = 0$. The input volume velocity Q_0 is known, and Q_{L2} can be calculated. Using the four-pole matrix (10), Q_{L1} can be obtained from equation (8).

Lumped Parameter Model In The Time Domain

This case is analogous to what is called in vibrations of a base-excited oscillator. The same diagram as Figure 1 can be used. The volume change dV in the volume attached to the pipe is now,

$$dV = \xi A - \int_0^t Q_0 dt \, . \tag{11}$$

Thus,

$$P = -\rho_0 C^2 \frac{\xi A}{V} + \frac{\rho_0 C^2}{V} \int_0^t Q_0 dt, \tag{12}$$

and equation of motion becomes,

$$LA\rho_0 \ddot{\xi} + D\dot{\xi} + \frac{\rho_0 C^2 A^2}{V} \xi = \frac{\rho_0 A}{V} \int_0^t Q_0 dt \, . \tag{13}$$

This equation can now be solved if the volume flow rate at the entrance is known as a function of time. The Runge-Kutta method can be used to solve the differential equations in this case.

Continuous Model In Four-Pole Form

In the case of a pipe which is long relative to the shortest acoustic wavelength of interest, the gas in the pipe can no longer be considered incompressible. The one-dimensional wave equation for the pipe has to be solved to obtain the four-pole relations. For the cavity, the four-pole matrix is exactly the same as in equation (5). For the pipe, we need to solve the wave equation,

$$\frac{\partial^2 \xi}{\partial t^2} + \gamma_1 \frac{\partial \xi}{\partial t} = C_0^2 \frac{\partial^2 \xi}{\partial x^2}, \tag{14}$$

where ξ is the particle displacement, $\gamma_1 = r_1/\zeta_0$ and r_1 = equivalent viscous coefficient [N sec/m^4]. The solution for harmonic waves is,

$$\xi(x,t) = A_1 e^{j(\omega t - k_1 x)} + B_1 e^{j(\omega t + k_1 x)}, \tag{15}$$

where k_1 = wave number. Substituting the solution into the equation of motion gives the expression of k_1 [1],

$$\gamma = jk_1 = a + jk, \quad \text{where } k = \frac{\omega}{C_0}, a = \frac{\gamma_1}{2C_0} \quad . \tag{16}$$

The pressure becomes,

$$P(x,t) = -\rho_0 C_0^2 \frac{\partial \xi}{\partial x}, Q(x,t) = A\dot{\xi} \, . \tag{17}$$

The volume velocity can be obtained by differentiating equation (15) with respect to time. The solution can be written as,

$$P(x,t) = \rho_0 C_0^2 \gamma [A_1 e^{-\gamma x} - B_1 e^{+\gamma x}] e^{j\omega t}, \; Q(x,t) = j\omega A [A_1 e^{-\gamma x} - B_1 e^{+\gamma x}] e^{j\omega t} \, . \tag{18,19}$$

Now, the boundary conditions at each end of the pipe can be applied to equations (18) and (19). These boundary conditions are: $P(0,t) = P_{L1} e^{j\omega t}, P(L,t) = P_{L2} e^{j\omega t}, Q(0,t) = Q_{L1} e^{j\omega t}, Q(L,t) = Q_{L2} e^{j\omega t}$. (20)

The result is,

$$\begin{Bmatrix} Q_{L1} \\ P_{L1} \end{Bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{Bmatrix} Q_{L2} \\ P_{L2} \end{Bmatrix}, \tag{21}$$

where,

$$b_1 = b_4 = \cosh(\gamma L), \; b_2 = \frac{j\omega A}{\rho_0 C_0^2 \gamma} \sinh(\gamma L), \; b_3 = \frac{\rho_0 C_0^2 \gamma}{j\omega A} \sinh(\gamma L) \quad . \tag{22}$$

As in the case of the lumped parameter formulation in four-pole form, the volume velocity at the entrance of the pipe can be obtained and will be compared.

CONTINUOUS SYSTEM MODEL IN THE TIME DOMAIN

Governing Equation

The equation of motion for gas particles in uniform or piecewise uniform pipes is the same as equation (14). If the ends of the pipe are defined by x=0,L, the boundary conditions are,

$$\frac{\partial \xi}{\partial x}(L,t)=0,\ \ \rho_0 A\,\xi(0,t)=\int_0^t \rho_0 Q_{L1}dt\ . \tag{23}$$

To solve this problem, we substitute, $\xi(x,t)=\xi(o,t)+\eta(x,t),$ (24)

The equation of motion becomes, $\frac{\partial^2 \eta}{\partial x^2}-\frac{\gamma_1}{C_0^2}\frac{\partial \eta}{\partial t}-\frac{1}{C_0^2}\frac{\partial^2 \eta}{\partial t^2}=\frac{1}{C_0^2}\frac{\partial^2 \xi(0,t)}{\partial t^2},$ (25)

and the boundary conditions become, $\frac{\partial \eta}{\partial x}(L,t)=0,\ \ \eta(0,t)=0\ .$ (26,27)

Therefore, the original equation of motion has been transformed into an equation with homogeneous boundary conditions [2].

Modes and Natural Frequencies Of The Gas Column

Solving the homogeneous part of equation (25), the solution can be expressed in the form,

$$\eta(x,t)=\overline{\eta}(x)e^{j\omega t}. \tag{28}$$

Substituting this into the homogeneous undamped equation gives,

$$\frac{d^2\overline{\eta}}{dx^2}+\left(\frac{\omega}{C_0}\right)^2\overline{\eta}=0. \tag{29}$$

Its solution is, $\overline{\eta}(x)=A\sin\frac{\omega}{C_0}x+B\cos\frac{\omega}{C_0}x,$ (30)

Equations (26) and (27) become, $\frac{\partial \overline{\eta}}{\partial x}(L)=0,\ \ \overline{\eta}(0)=0\ .$ (31,32)

Applying equation (29) to the boundary conditions gives the natural frequencies and modes:

$$\omega_n=(2n-1)\frac{\pi}{2}\frac{C_0}{L},\ \ \overline{\eta}_n(x)=\sin\frac{\omega_n}{C_0}x\ . \tag{33,34}$$

Conversion Of Equivalent Viscous Damping From Model To Model

The wave equation can be solved by using the modal series expansion:

$$\overline{\eta}(x,t)=\sum_{n=1}^{\infty}q_n(t)\overline{\eta}_n(x)\ . \tag{35}$$

Making use of equation (28) and incorporating it into equation (25) gives,

$$\sum_{n=1}^{\infty}\left[\frac{d^2q_n}{dt^2}+\gamma_1\frac{dq_n}{dt}+\omega_n^2 q_n\right]\overline{\eta}_n(x)=-\frac{\partial^2\xi}{\partial t^2}(0,t)\ . \tag{36}$$

The damping term can be written as: $\gamma_1=2\zeta_n\omega_n\ .$ (37)

Therefore, the damping coefficient of each mode can be presented in terms of the viscous coefficient η_1 used in four-pole models, as, $\zeta_n=\frac{\gamma_1}{2\omega_n}\ .$ (38)

Solution By Modal Series Expansion

Using orthogonality of the eigenfunctions, we can multiply equation (36) by the eigenfunctions and integrate. This allows us to remove the summation, and equation (36) becomes,

$$\frac{d^2q_n}{dt^2}+2\zeta_n\omega_n\frac{dq_n}{dt}+\omega_n^2q_n=-\frac{1}{N_n}\frac{\partial^2\xi(0,t)}{\partial t^2}\int_0^L\overline{\eta}_n(x)dx, \tag{39}$$

where, $N_n=\int_0^L\overline{\eta}_n^2(x)dx.$ (40)

To solve equation (39) as applied to the volume-pipe system, one more expression is needed. The pressure expressions at the boundary of the cavity and the pipe must be the same. $P_L=P_{L1},$ (41)

$$P_L=\frac{C_0^2}{V}\int_0^t\rho_0Q_0dt+\frac{\rho_0C_0^2A}{V}\xi(0,t), \tag{42}$$

$$P_{L1} = -\rho_0 C_0^2 \left[\frac{\partial \xi(x,t)}{\partial x} \right]_{x=0} \tag{43}$$

Now, the whole system of equations can be solved. The volume velocity can be obtained by differentiation:

$$Q(x,t) = A\dot{\xi}(x,t) = A[\dot{\xi}(0,t) + \dot{\eta}(x,t)] \tag{44}$$

RESULTS AND CONCLUSIONS

Figure 2 displays the relations between magnitude of volume velocity and input frequency. Figure 2 (a) shows the comparison of the four-pole approach and time domain technique, considering the system as a lumped model. The peak is demonstrated to be at the natural frequency of the lumped system. Figure 2 (b) is the same as Figure 2 (a), but considers the system to be continuous. Three natural modes were used for the continuous model in the time domain. The first peak of Figure 2 (b) corresponds to the previously obtained natural frequency. However, there are other peaks in the continuous model because the higher natural frequencies of the pipe are considered. Since three modes of the pipe were considered in the time domain plus a rigid body motion, four peaks including the peak of the lumped parameter system can be seen. Figures 3 and 4 show the variation of volume velocity with respect to time. The four-pole method gives only the steady-state response. Transient motions can be seen only when using the time domain approach. For small damping, more transient motion is expected (Figure 5).

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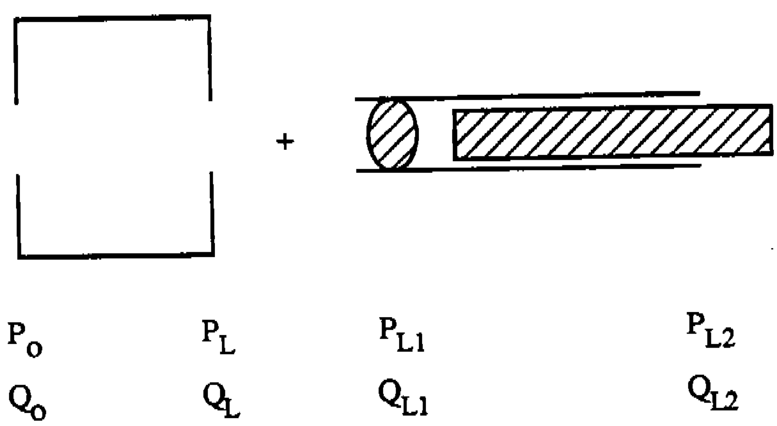


Figure 1. Free diagram of the gas in the pipe

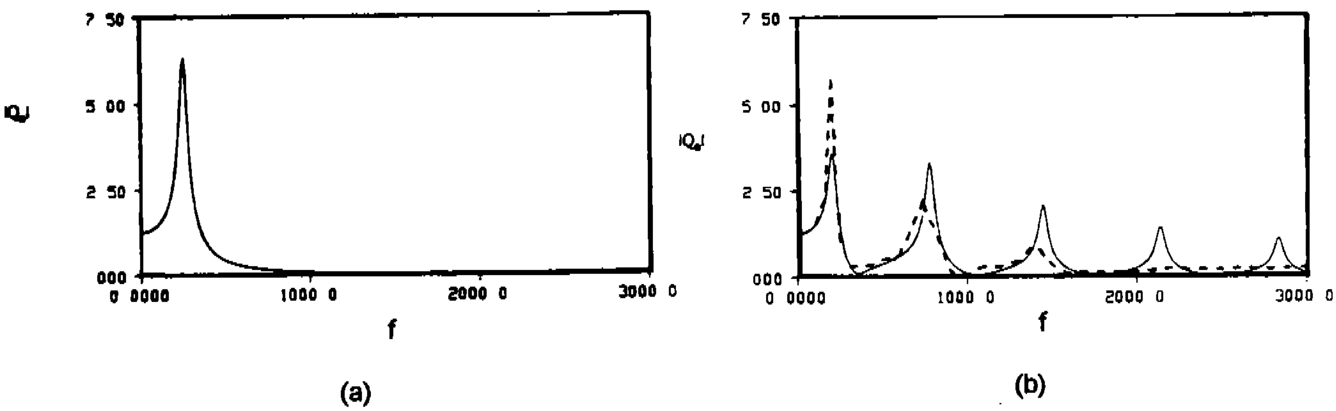


Figure 2. Magnitude of volume velocity: (a) Lumped model, only one peak can be seen, (b) continuous approach, using the four pole method, —, and the time domain approach, - - - . There was no attempt to adjust the damping.

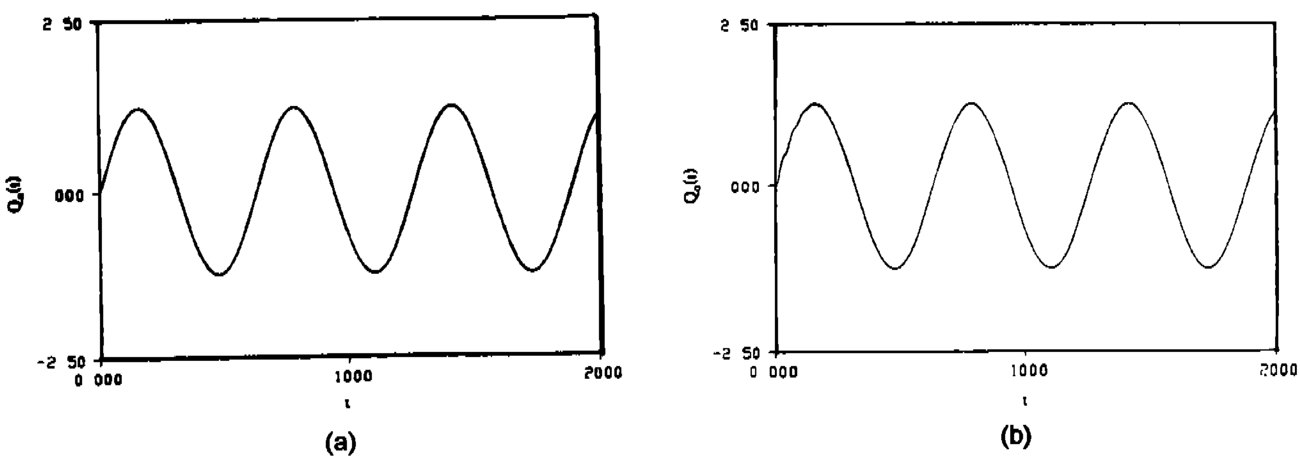
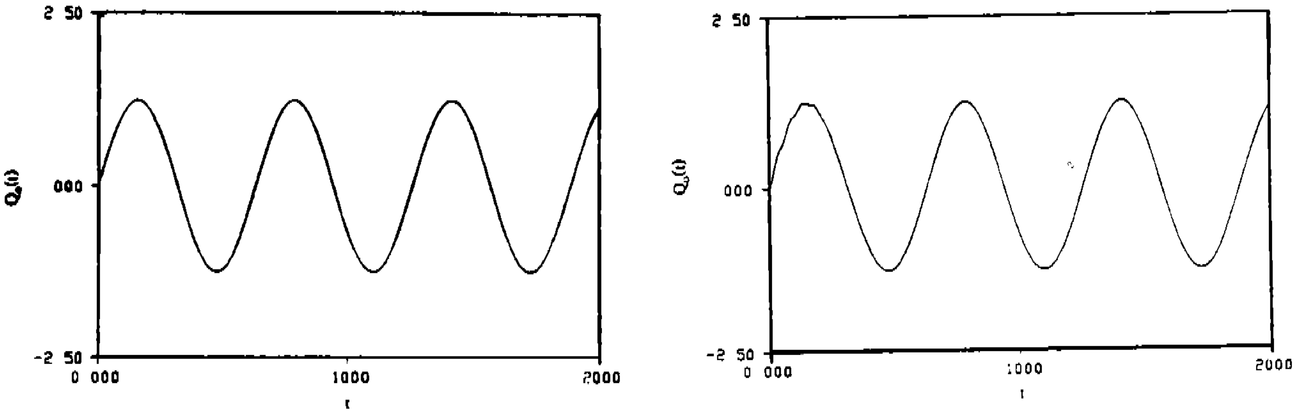
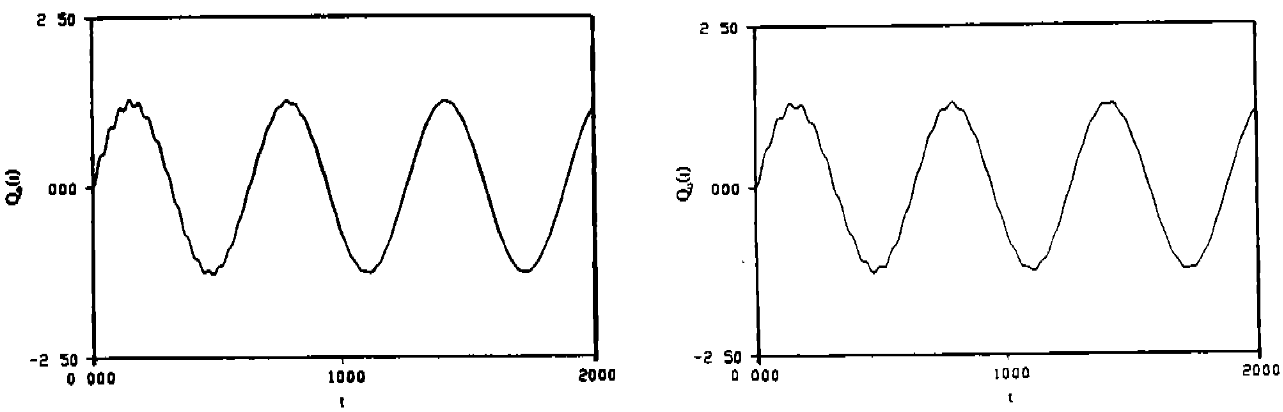


Figure 3. Volume velocity response with respect to time for w=628 rad/s of a lumped model: (a) using four pole method, (b) using time domain technique.



(a) (b)

Figure 4. Volume velocity response with respect to time of a continuous model: (a) with the four pole method, (b) using time domain technique.



(a) (b)

Figure 5. Volume velocity variation for low damping using time domain technique: (a) lumped model, (b) continuous model.